Lecture 16
Dynamic Programming

CS 161 Design and Analysis of Algorithms

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- [CLRS] Chapter 15

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- [GT]: Chapter 12
- [CLRS] Chapter 15
- [Kleinberg and Tardos], Chapter 6

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    - This requires careful indexing of subproblems

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	D&C / Recursion	Memoized Recursion	Dynamic Programming
Basic approach	recursion	recursion	iteration
Use of recurrence	top-down	top-down	bottom-up
Store subproblem solutions	No	Yes	Yes
Space needed for stack	Yes	Yes	No

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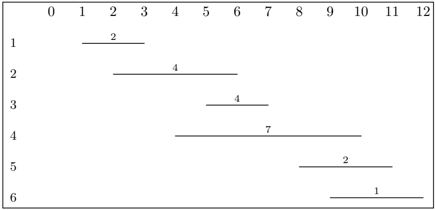
Problem: Weighted interval scheduling

- **Input:** Collection of n Intervals represented by Start Time, Finish Time, and Value: $(s(j), f(j), v(j))$.
- **Problem:** Find a non-overlapping set of intervals that maximizes the total value.
- **Example:**

j	$s(j)$	$f(j)$	$v(j)$
1	1	3	2
2	2	6	4
3	5	7	4
4	4	10	7
5	8	11	2
6	9	12	1



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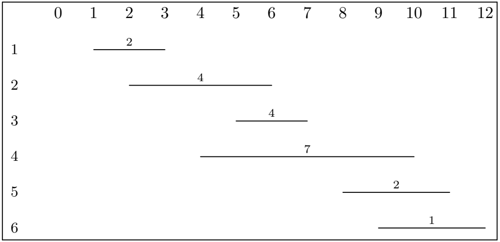

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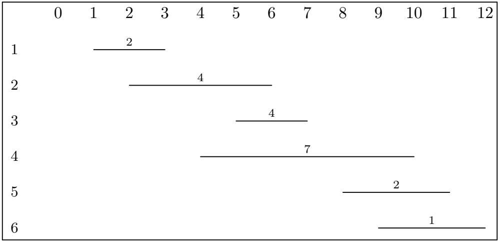

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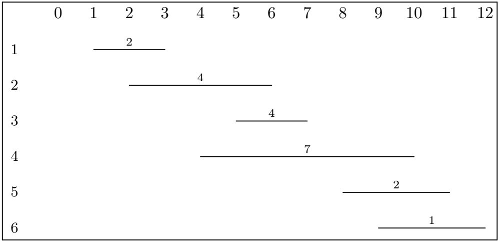

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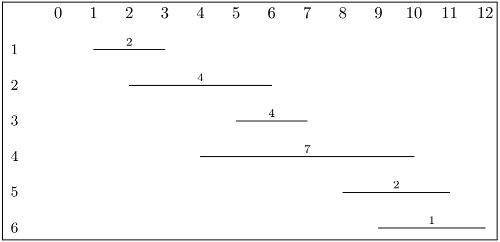

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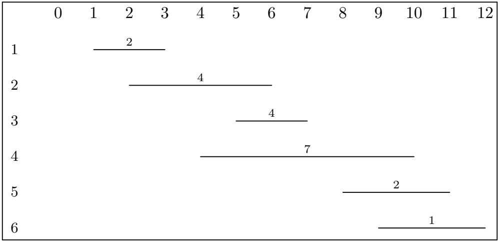

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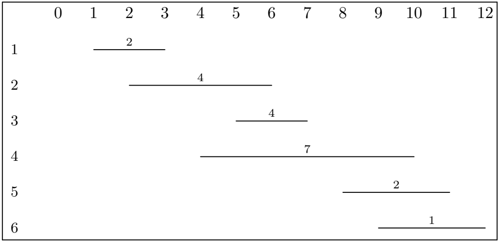

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def OPT(j):
    if j = 0:  return 0
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- Hence, $O(n)$ calls.

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- Run a post-processing step that uses this additional information

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```
def Iterative_OPT:
   M[0] = 0
   for j = 1 to n:
      if v(j)+M[p(j)] > M[j-1]:
         M[j] = v(j)+M[p(j)]
         keep[j] = True
      else:
         M[j] = M[j-1]
         keep[j] = False
```

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```
def PrintSolution(j):
   if j = 0:  return;
   if keep[j]:
      PrintSolution(p(j))
      print(j)
   else:
      PrintSolution(j-1)

PrintSolution(n)
```

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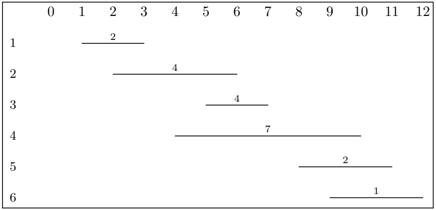

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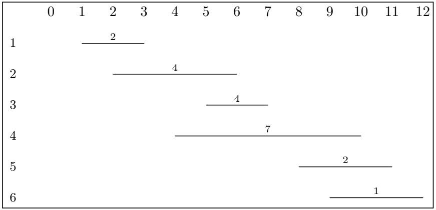

	0	1	2	3	4	5	6
`M:`	0	2	4	6	9	9	9
`keep:`		T	T	T	T	F	F

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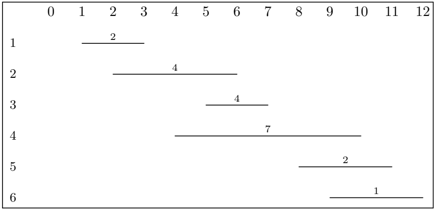
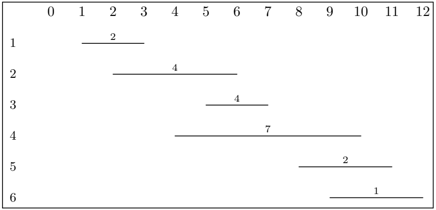


	0	1	2	3	4	5	6
`M:`	0	2	4	6	9	9	9
`keep:`		T	T	T	T	F	F

Selected intervals: $\{1, 4\}$.

Our example

j	$s(j)$	$f(j)$	$v(j)$	$p(j)$
1	1	3	2	0
2	2	6	4	0
3	5	7	4	1
4	4	10	7	1
5	8	11	2	3
6	9	12	1	3



	0	1	2	3	4	5	6
`M:`	0	2	4	6	9	9	9
`keep:`		T	T	T	T	F	F

Selected intervals: $\{1, 4\}$.

The array M contains the solutions of the subproblems. We will refer to this as the memoization table

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We saw this in the case of the weighted interval scheduling problem.

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Here, "smaller" means "earlier in the ordering"

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5. Recurrence: $M(j) = \max(v(j) + M(p(j)), M(j-1))$ for $j \geq 1$. Here, $p(j)$ is a precomputed function defined by

$$p(j) = \begin{cases} \text{The highest-numbered interval } i < j \text{ that does not} \\ \text{overlap interval } j \text{ if such an interval exists} \\ 0 \text{ otherwise} \end{cases}$$

Truck loading problem

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Dynamic Programming Solution: Basic Idea

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We will express this more formally on the next slide.

Solution: Expressed as recurrence equation

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- Let $\text{OPT}(i, j)$ be the maximum weight we can get by loading from boxes 1 through i, up to the weight limit j.

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$$\mathrm{OPT}(i, j) = \max\left(w_i + \mathrm{OPT}(i-1, j-w_i), \mathrm{OPT}(i-1, j)\right)$$

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- Note that if $w_i > j$, we can't use box i, so only the second choice is available.

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- Note that if $w_i > j$, we can't use box i, so only the second choice is available.
- This recurrence equation gives us the dynamic programming solution (specified on next slide)

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1. Subproblem domain: $\{0, \ldots, n\} \times \{0, \ldots, W\}$

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4. Initial values:

$$\begin{array}{rcll} \mathrm{OPT}(i, 0) & = & 0 & \text{for all } i \geq 0 \\ \mathrm{OPT}(0, j) & = & 0 & \text{for all } j \geq 0 \end{array}$$

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5. Recurrence:

$$\mathrm{OPT}(i,j) = \begin{cases} \max\left(w_i + \mathrm{OPT}(i-1, j-w_i), \mathrm{OPT}(i-1,j)\right) & \text{if } w_i \leq j \\ \mathrm{OPT}(i-1,j) & \text{if } w_i > j \end{cases}$$

Truck Loading Problem DP Pseudocode: compute OPT Matrix

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```
def compute_opt_matrix(w):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if w[i] > j:
           OPT[i,j] = OPT[i-1,j]
        else:
           OPT[i,j] = max(w[i] + OPT[i-1,j-w[i]], OPT[i-1,j])
  return OPT
```

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        else:
           OPT[i,j] = max(w[i] + OPT[i-1,j-w[i]], OPT[i-1,j])
  return OPT
```

This tells us the maximum possible weight, but we need to also compute which boxes to load to achieve this maximum weight . . .

Truck Loading Problem DP Pseudocode: compute choice of boxes

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Introduce an new array `keep[i,j]`, which tells us whether we keep box i when we solve the subproblem with i boxes and capacity j.

Truck Loading Problem DP Pseudocode: compute choice of boxes

Introduce an new array `keep[i,j]`, which tells us whether we keep box *i* when we solve the subproblem with *i* boxes and capacity *j*.

```
def compute_opt_strategy(w):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if (w[i] > j) or (w[i] + OPT[i-1,j-w[i]] <= OPT[i-1,j])
           OPT[i,j] = OPT[i-1,j]
           keep[i,j] = False
        else:
           OPT[i,j] = w[i] + OPT[i-1,j-w[i]]
           keep[i,j] = True
     return (OPT,keep)
```

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```
def compute_opt_strategy(w):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
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           OPT[i,j] = OPT[i-1,j]
           keep[i,j] = False
        else:
           OPT[i,j] = w[i] + OPT[i-1,j-w[i]]
           keep[i,j] = True
     return (OPT,keep)
```

Running time: $O(n \cdot W)$

Truck Loading Problem DP Pseudocode: compute choice of boxes [continued]

Truck Loading Problem DP Pseudocode: compute choice of boxes [continued]

```
def print_solution(OPT,keep,i,j):
   if i == 0:  return
   if keep[i,j]:
      print_solution(OPT,keep,i-1,j-w[i])
      print (i)
   else:
      print_solution(OPT,keep,i-1,j)

// Main program starts here
(OPT,keep) = compute_opt_strategy(w)
print_solution(OPT,keep,n,W)
```

Truck Loading Problem Example

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

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j

i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

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j

		0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -	0 -
i	1	0 -	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
	2	0 -	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
	3	0 -	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

		0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
i	1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
	2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
	3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

i

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Keep box 3.

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

i		0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
	1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
	2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
	3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Keep box 3. $\Rightarrow i = 2, j = 12 - 7 = 5$

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

i

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Keep box 3. $\Rightarrow i = 2, j = 12 - 7 = 5$

Keep box 2.

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Keep box 3. $\Rightarrow i = 2, j = 12 - 7 = 5$

Keep box 2. $\Rightarrow i = 1, j = 5 - 4 = 1$

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity $= 12$.

j

i

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
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Solution:

Maximum weight $= 11$

Keep box 3. $\Rightarrow i = 2, j = 12 - 7 = 5$

Keep box 2. $\Rightarrow i = 1, j = 5 - 4 = 1$

Do not keep box 1.

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

j

		0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –	0 –
i	1	0 –	0 F	0 F	0 F	0 F	0 F	0 F	0 F	0 F	9 T	9 T	9 T	9 T
	2	0 –	0 F	0 F	0 F	4 T	4 T	4 T	4 T	4 T	9 F	9 F	9 F	9 F
	3	0 –	0 F	0 F	0 F	4 F	4 F	4 F	7 T	7 T	9 F	9 F	11 T	11 T

Solution:

Maximum weight = 11

Keep box 3. $\Rightarrow i = 2, j = 12 - 7 = 5$
Keep box 2. $\Rightarrow i = 1, j = 5 - 4 = 1$
Do not keep box 1. $\Rightarrow i = 0, j = 1$

Truck Loading Problem Example

3 boxes with weights 9, 4, and 7. Truck capacity = 12.

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    - $W = 100$
    - Item 1: $w_1 = 20$, $v_1 = 80$
    - Item 2: $w_2 = 90$, $v_2 = 90$.

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5. Recurrence:

$$\mathrm{OPT}(i,j) = \begin{cases} \max\left(v_i + \mathrm{OPT}(i-1, j-w_i), \mathrm{OPT}(i-1,j)\right) & \text{if } w_i \leq j \\ \mathrm{OPT}(i-1,j) & \text{if } w_i > j \end{cases}$$

Pseudocode for DP Solution to 0/1 Knapsack Problem

```
def compute_opt_strategy(w,v):
  for i = 0 to n:  OPT[i,0] = 0
  for j = 0 to W: OPT[0,j] = 0
  for i = 1 to n:
     for j = 1 to W:
        if (w[i] > j) or (v[i] + OPT[i-1,j-w[i]] <= OPT[i-1,j])
           OPT[i,j] = OPT[i-1,j]
           keep[i,j] = False
        else:
           OPT[i,j] = v[i] + OPT[i-1,j-w[i]]
           keep[i,j] = True
     return (OPT,keep)
```

Pseudocode for DP Solution to 0/1 Knapsack Problem [continued]

```
def print_solution(OPT,keep,i,j):
   if i == 0:  return
   if keep[i,j]:
      print_solution(OPT,keep,i-1,j-w[i])
      print (i)
   else:
      print_solution(OPT,keep,i-1,j)

// Main program starts here
(OPT,keep) = compute_opt_strategy(w,v)
print_solution(OPT,keep,n,W)
```

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Parenthesization Matters

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Example:

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- Suppose the left subchain stops at A_k: $(A_i \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A_j)$

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```
def optMatrixChain(d):
    for i = 1 to n:
        M[i,i] = 0
    for len = 2 to n:
        for i = 1 to n - len + 1:
            j = i + len - 1
            M[i,j] = $+\infty$
            for k = i to j-1:
                x = M[i,k] + M[k+1,j] + d[i-1]*d[k]*d[j]
                if x < M[i,j]:
                    M[i,j] = x
    return M
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0 —	750 1	3750 2	35750 2	41750 2	46750 2	56500 2	1
	0 —	4500 2	37500 2	41500 2	47000 2	56925 2	2
		0 —	30000 3	40000 4	44000 5	53400 6	3
			0 —	120000 4	168000 5	214000 5	4
				0 —	80000 5	131600 5	5
					0 —	37600 6	6
						0 —	7

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$$(A_1 \times A_2) \times ((A_3 \times A_4 \times A_5 \times A_6) \times A_7)$$

Solution to our example

$A_1 : 10 \times 15$
$A_2 : 15 \times 5$
$A_3 : 5 \times 60$
$A_4 : 60 \times 100$
$A_5 : 100 \times 20$
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$d_0 = 10$
$d_1 = 15$
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Suppose we have the following data values and frequency values:

i	Data	p_i
1	A	.26
2	B	.06
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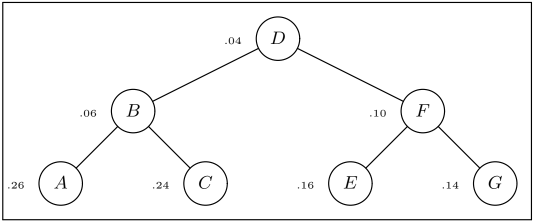

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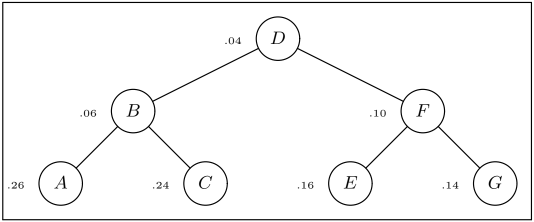

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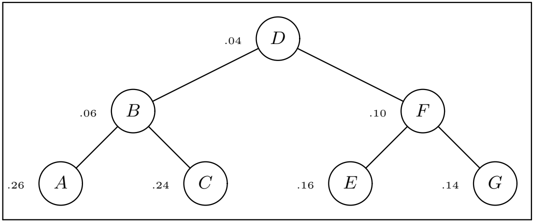

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- The generalization to allowing unsuccessful searches is discussed in [CLRS].

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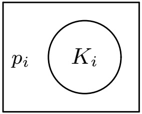

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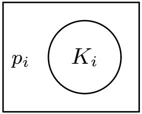

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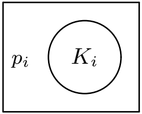

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We need to develop a recurrence equation . . .

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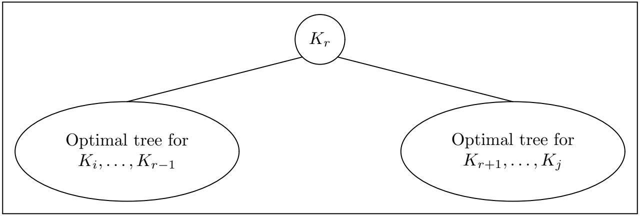

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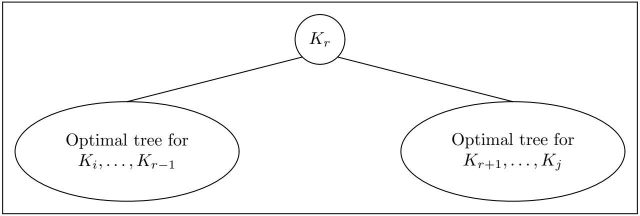

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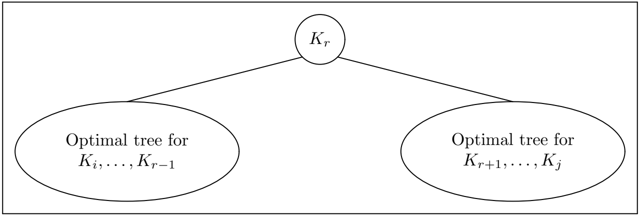

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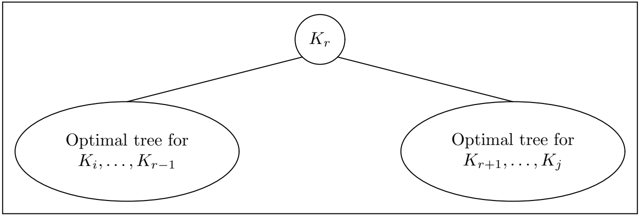

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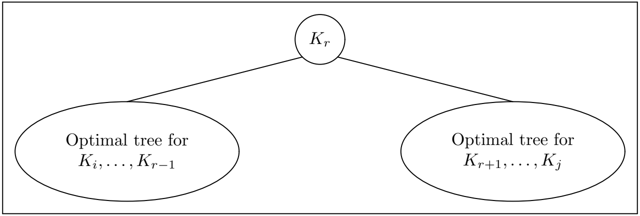

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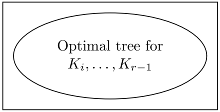
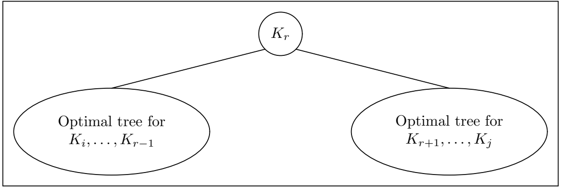


Optimal tree for $K_i, \ldots, K_{r-1}$



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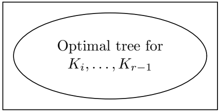

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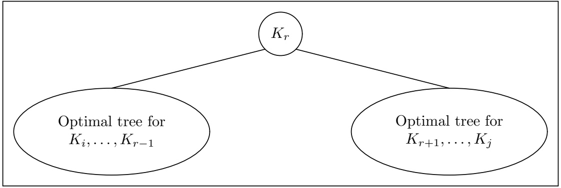

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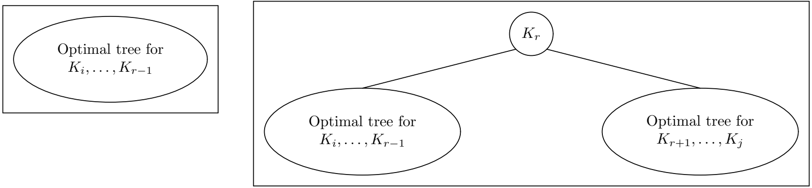

Observation:

- The weighted cost of the optimal tree on $K_i, \ldots, K_{r-1}$ is $E(i, r-1)$.

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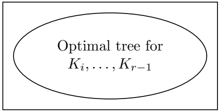

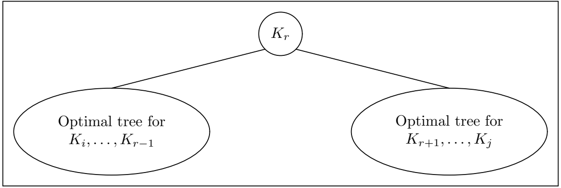

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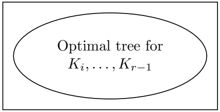

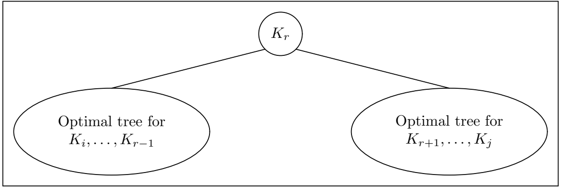

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- So the total weighted cost of the nodes $K_i, \ldots, K_{r-1}$ in the tree rooted at K_r is is

$$E(i, r-1) + p_i + p_{i+1} + \ldots + p_{r-1}.$$

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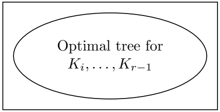

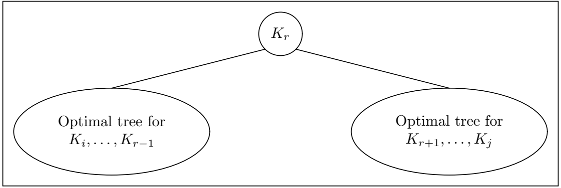

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- Similarly, the total weighted cost of the nodes $K_{r+1}, \ldots, K_j$ is

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Develop recurrence equation [continued]

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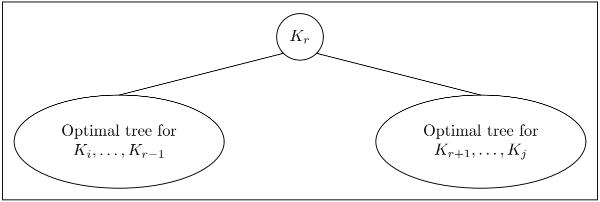

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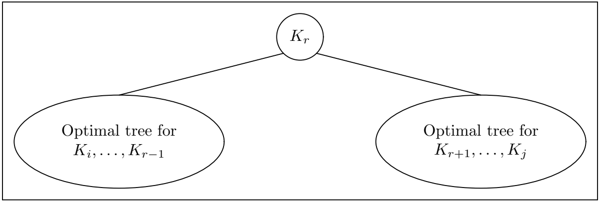

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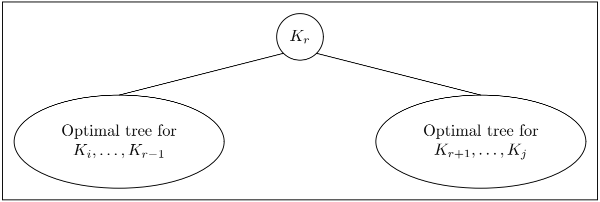

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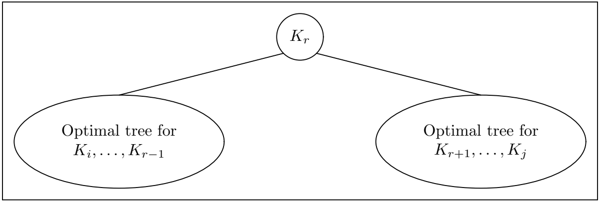

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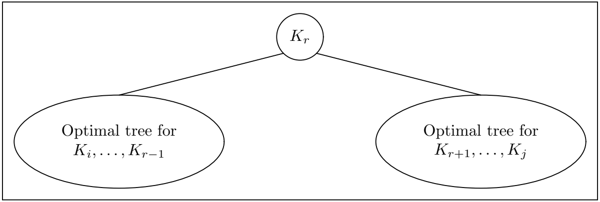

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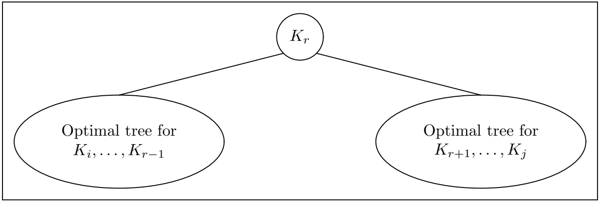

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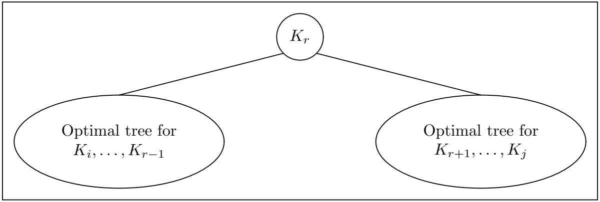

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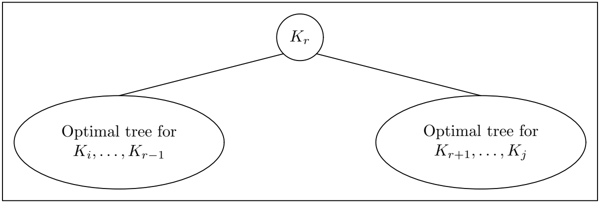

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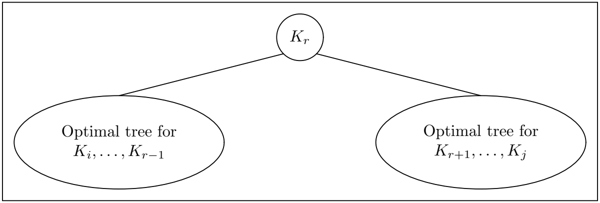

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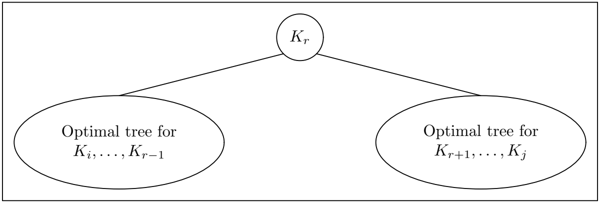

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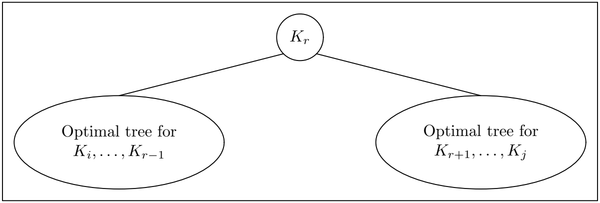

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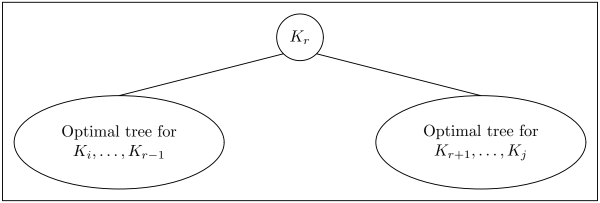

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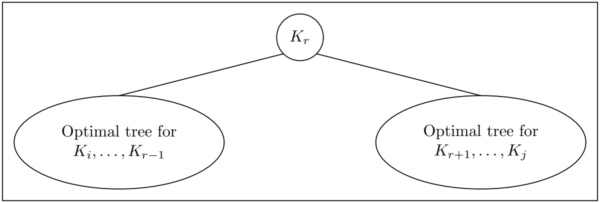

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  ```
  for i = 1 to n+1:
      W[i,i-1] = 0
      for j = i to n
          W[i,j] = W[i,j-1] + p[j]
  ```

Code to compute the cost of the optimal tree

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```
def OptimalTreeCost(p):
    for i = 1 to n+1:
        E[i,i-1] = 0
        W[i,i-1] = 0
        for j = i to n
            W[i,j] = W[i,j-1] + p[j]
    for size = 1 to n:
        for i = 1 to n - size + 1 do
            j = i + size - 1
            E[i,j] = $+\infty$;
            for r = i to j:
                x = E[i,r-1] + E[r+1,j] + W[i,j]
                if x $<$ E[i,j]:
                    E[i,j] = x
    return(E)
```

Modified code to compute the root of each optimal subtree

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To compute the optimal tree, we need to store the root of each optimal subtree. We compute a second array `root` which tells us the best root of the tree for the keys $K_i, \ldots, K_j$:

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            for r = i to j:
                x = E[i,r-1] + E[r+1,j] + W[i,j]
                if x $<$ E[i,j]:
                    E[i,j] = x
                    root[i,j] = r
    return(E,root)
```

Code generate the optimal tree

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Once we have computed the arrays `E` and `root`, the following pseudocode computes the optimal binary tree:

```
def OptimalTree(root,keys):

    // keys is the array of key values, indexed from 1 to n

    def buildTree(i,j):
        if j < i :  return null
        r = root[i,j]
        node = new binary tree node
        node.key = keys[r]
        node.leftchild = buildTree(i,r-1)
        node.rightchild = buildTree(r+1,j)
        return node

    return buildTree(1,n)
```

Solution to our example

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i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

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i	Data	p_i
1	*A*	.26
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j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
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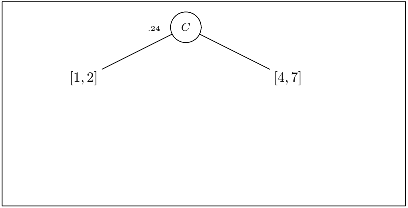

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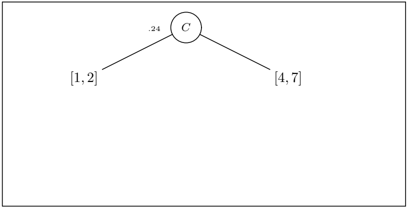

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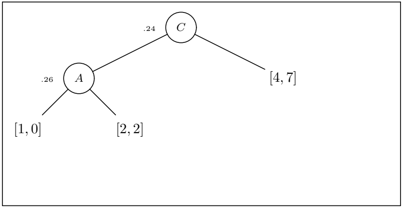
							0 —	8

i



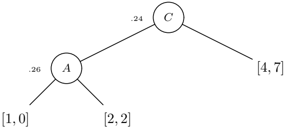
Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

i

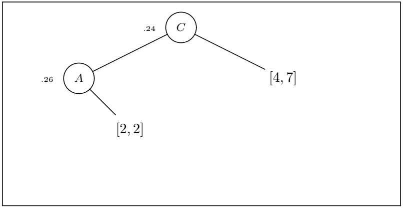


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

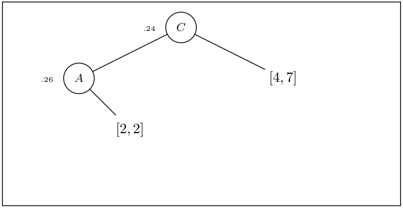


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

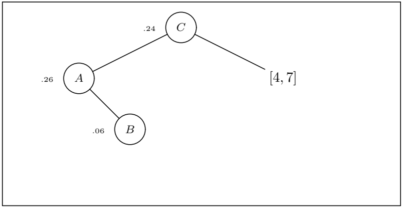


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

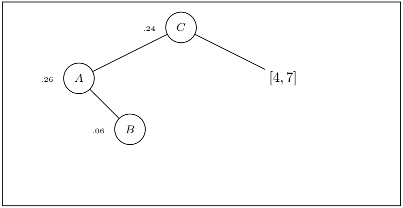


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

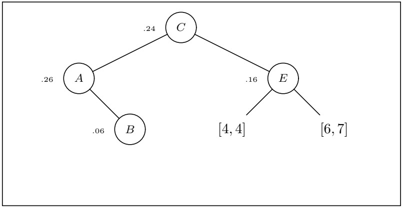


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

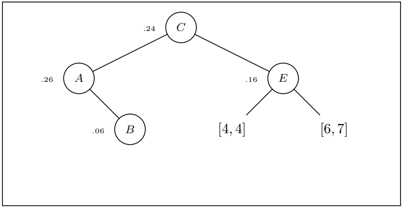


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

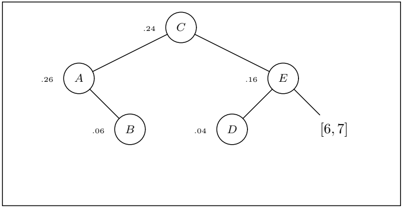
0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8



Solution to our example

i	Data	p_i
1	*A*	.26
2	*B*	.06
3	*C*	.24
4	*D*	.04
5	*E*	.16
6	*F*	.10
7	*G*	.14

i \ *j*	0	1	2	3	4	5	6	7
1	0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3
2		0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5
3			0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5
4				0 —	0.04 4	0.24 5	0.44 5	0.82 5
5					0 —	0.16 5	0.36 5	0.70 6
6						0 —	0.10 6	0.34 7
7							0 —	0.14 7
8								0 —

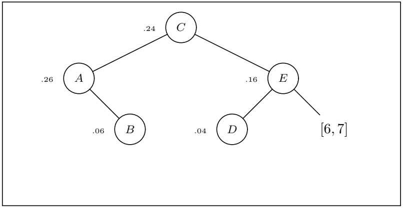


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

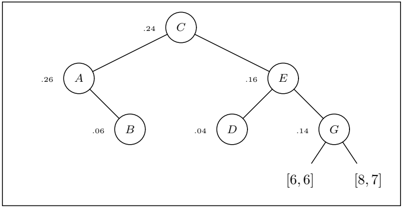


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

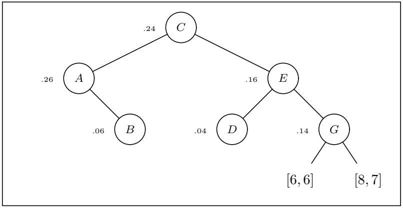


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

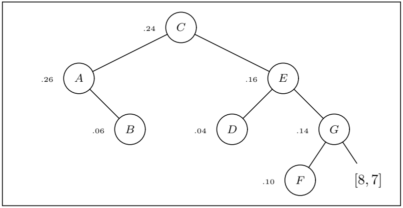


Solution to our example

i	Data	p_i
1	*A*	.26
2	*B*	.06
3	*C*	.24
4	*D*	.04
5	*E*	.16
6	*F*	.10
7	*G*	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

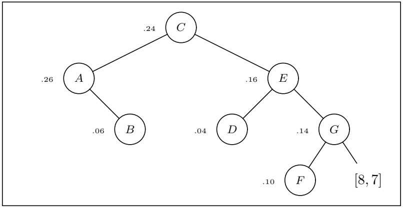


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

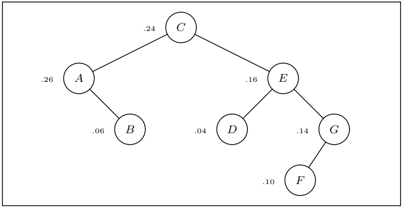


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

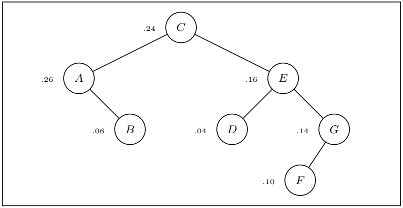


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8

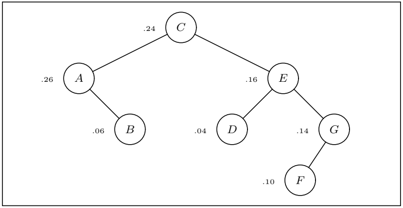


Solution to our example

i	Data	p_i
1	A	.26
2	B	.06
3	C	.24
4	D	.04
5	E	.16
6	F	.10
7	G	.14

j

0	1	2	3	4	5	6	7	i
0 —	0.26 1	0.38 1	0.92 1	1.02 3	1.38 3	1.68 3	2.20 3	1
	0 —	0.06 2	0.36 3	0.44 3	0.80 3	1.10 3	1.52 5	2
		0 —	0.24 3	0.32 3	0.68 3	0.96 5	1.34 5	3
			0 —	0.04 4	0.24 5	0.44 5	0.82 5	4
				0 —	0.16 5	0.36 5	0.70 6	5
					0 —	0.10 6	0.34 7	6
						0 —	0.14 7	7
							0 —	8



Weighted lookup cost = 2.20